Environmental management: analytical approximate solutions to the problem of detecting optimal random audit schemes *

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Abstract. In the contest of environmental management, the problem of minimizing the expected cost due to random checking processes and a possible failure is here addressed. Non-homogeneous Poisson checking processes with continuous non-decreasing intensity are considered, leading to the explicit detection of the sub-optimal solution for exponential or uniform failure density functions. The dynamic of the optimal solution is then analized using the phase-diagram tool.
Keywords. Environmental management; audit scheme; random inspections; non-homogeneous Poisson checking process; optimal control; exponential failure density function; uniform failure density function.

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1 Introduction

In the past few years the number of standards in the field of environmental techniques and environment management has augmented. One of the main meaningful step of the European Council regulation Eco Management and Audit Scheme (EMAS) sets the tasks of the involved actors. Between them, a very important role is played by verifiers: in fact monitoring and reporting is a crucial aspect for checking the fulfillment of the environmental agreements. All the elements of the management system has to been audited over a three year cycle. In that period the process of validation of the Environmental Statement requires a particularly detailed process of checking: one of the main efforts is addressed to verify that the Environmental Statement gives a realistic picture of the organization performance. This aspect is directly connected with the increasing demand of consumers and shareholders of safe and environmentally-friendly

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products: think, for example, to the actual question of BSE (Bovine Spongiform Encephalopathy) test on beef or to the international debate on GMO's (Genetically Modified Organisms). Even if EMAS is a voluntary initiative designed to improve organizations' environmental performance, one of its aim is represented by recognizing and rewarding those companies. Credibility and recognition of the involved organizations are enhanced by ensuring checking processes made by independent environmental verifiers. Moreover, the company's top management is required to check periodically the agreement and consistency of the environmental management system with the objectives stated in the environmental policy and program. In this way, incidental mistakes are analyzed and immediately removed. This improvement of the environmental performance through internal audits can be achieved with random inspection schedules. In fact, it is plausible to assume that an inconsistency begins possibly at a random epoch and continues, unless it is detected. In the process of reviewing it is important to record the possible mistake and the person responsible of it: in this way, the immediate and remote causes of the inconsistency can be removed. A deterministic checking process is predictable, so the incorrectness could be masked and the cause could be hidden. The effects of the corrective actions would be bounded, because of the untraceability of the responsible of the mistake. Moreover, the inspections are done by the organization's top manager who doesn't stay idle between any two consecutive inspections, but he is engaged in activities requiring a random time. Then, one might be able to control the random interval between two consecutive audits, without being able to determine it precisely. This goal could be achieved by accepting some operations and refusing others, on the basis of some prior information on their time-length probability distribution. The paper is organized as follows. In Section 2 we first set the definition of the problem of detecting the optimal checking schedules of a system subject to failure and then review some preliminary results in that field. Section 3 is devoted to the analysis of the optimal solutions of the related problem where random inspection schedules with non-decreasing intensity are assumed, in the cases of exponential. The next section analyzes the case of uniform failure density function. Finally Section 5 is devoted to some conclusive observations.

2 Problem setting

An inspection schedule is an increasing sequence of times at which different checks have to be done. We call *failure* a situation in which the system does not verify compliance with some rules, such as environmental management standards (UNI EN ISO 14001 and EMAS). When the system is assumed to *work* then the required standards are achieved.

If we denote with $t = 0$ the time at which the system starts working, then the problem of detecting the possible first failure is studied in the period $[0, t_1]$ ($t_1 > 0$). We assume that

- each inspection does not influence the state of the system and it requires only a negligible time;

– the system cannot fail while it is inspected;
– each check induces a fixed cost c_0 ($c_0 > 0$);
– the checking process ends
  - just after finding a failure, if this occurs by the time t_1;
  - at the first check following the time t_1.

The first system failure occurs at a time T, where T is a positive random variable with probability distribution function F and density f. We have supposed that the first failure is relevant if and only if it occurs by the fixed finite time t_1. That fixed time sets the interest interval, which may vary in function of the reference audit scheme: for instance, in the case of EMAS, $t_1 = 36$ months. Let X be the failure detection delay: if $T \in [0, t_1]$ then $T + X$ represents the time of discovery of the first failure. In this case, the inspection procedure immediately ends. Otherwise, the checking process stops after the first check following the time t_1: this is why the failure $T > t_1$ need not be discovered. Any delay from a failure to its detection generates a cost which is supposed a linear function of the delay. Thus the total cost due to the inspections and the failure is given by

$$C(M, X, T) = c_0 M + cX\chi_{T \le t_1}$$

where M denotes the number of inspections and $\chi_{T \le t_1}$ is the indicator function of the event $T \le t_1$. Let S be an inspection plan, that is

$$S = \{Y_k : k \ge 1\} \qquad 0 < Y_k < Y_{k+1},\ k \ge 1.$$

The last check is made at the time Y_M where M satisfies the following definition

$$M = M(S, T) = \min\{k : Y_k > T \wedge t_1, Y_k \in S\}$$

where $a \wedge b$ denotes the minimum between a and b. Our aim is devoted to determine a checking schedule S minimizing the expected total cost resulting from the inspections and the possible first failure. This problem is related to the analysis proposed by Ferretti and Viscolani ([3]) in the particular cases of homogeneous Poisson and linear Poisson checking schedules: while in the first case the explicit determination of the optimal solution has been proposed, in the second case an approximation of the original problem is discussed and the unique optimal solution is characterized in terms of Kuhn-Tucker conditions. Here we consider the more general case in which the number of checks during an interval $[0, t]$ is described by the Poisson checking process $N(t)$. In other words, let $N(t)$ denote the Poisson checking schedule with non-decreasing intensity $n(t) \ge 0$, where

$$n(t) = \begin{cases} n(t), & \text{if } 0 \le t \le t_1 \\ n(t_1), & \text{if } t > t_1 \end{cases}$$

and denote by $S_P(n)$ the related checking program

$$S_P(n) = \{Y_1, Y_2, \ldots, Y_k, \ldots\}$$

in which Y_k indicates the occurrence time of the k-th event concerning $N(t)$.

The expected total cost resulting from the inspections and the first possible failure is given by

$$E(C) = E[c_0 M + cX\chi_{T\leq t_1}] = E_T[c_0 E(M|T) + cE(X|T)\chi_{T\leq t_1}].$$

Our aim is to determine a checking program S minimizing $E(C)$. By using the monotonicity hypothesis on $n(t)$ and the assumption on the functional form of the cost due to the failure detection delay, it is possible to define a new problem (see Viscolani ([5])) with objective function $A(C)$ for which

$$E(C) \leq A(C)$$

and

$$A(C) = \int_0^{t_1} \left[c_0(1 + x_2(t)) + \frac{c}{x_1(t)} \right] f(t)dt + \int_{t_1}^{+\infty} c_0(1 + x_2(t_1))f(t)dt$$

where

$$x_1(t) = n(t), \qquad x_2(t) = \int_0^t n(w)dw.$$

In this way, it will be possible to determine the analytical approximate solution of the problem of detecting the optimal random audit scheme in the framework of the optimal control theory. In fact, in this case the control function is associated to the rate of growth of the checking intensity:

$$u(t) = \dot{x_1}(t).$$

Moreover, it is assumed that

a) the rate of growth of the checking intensity is upper bounded: $u(t) \leq \bar{u} \quad (\bar{u} > 0)$;
b) $x_1(0) = a \quad (a > 0)$.

The problem of determine an optimal audit scheme S minimizing the expected total cost $E(C)$ is replaced by the problem of minimizing its upper bound $A(C)$, denoted by *RAS* (*Random Audit Scheme*) which may be formalized such as follows:

$$RAS: \min_u A(C(u)) = c_0 + \int_0^{t_1} \left[c_0 x_2(t) f(t) + \frac{c}{x_1(t)} f(t) + c_0 x_1(t)(1 - F(t_1)) \right] dt$$

subject to the following constraints

$$\begin{array}{l} \dot{x}_1(t) = u(t), \;\; x_1(0) = a, \, a > 0, \\ \dot{x}_2(t) = x_1(t), \, x_2(0) = 0, \\ u(t) \in [0, \overline{u}], \quad u > 0 \end{array}$$

where x_1 and x_2 are the state variables which have been set equal to

$$x_1(t) = n(t), \qquad x_2(t) = \int_0^t n(w)dw.$$

This problem admits an optimal solution which is completely characterized by Pontryagin's Maximum Principle (see Seierstadt, Sydsaeter ([4]) and Viscolani ([5]): in fact, the result easily follows by setting $l(X) = cX$). The Hamiltonian function $H = H(\underline{x}, u, \underline{p}, t)$ of the problem RAS is

$$H(\underline{x}, u, \underline{p}, t) = -p_0 \left[c_0 x_2 f + \frac{c}{x_1} f + c_0(1 - F(t_1))x_1 \right] + p_1 u + p_2 x_1$$

where $\underline{p} = (p_0, p_1, p_2)$ and p_1, p_2 are the adjoint functions and $p_0 \in \{0, 1\}$ is a constant, $\underline{x} = (x_1, x_2)$. The Pontryagin's necessary and (in this problem) sufficient conditions assert

$$p_1(t_1) = 0, \tag{1}$$
$$p_2(t_1) = 0, \tag{2}$$
$$p_0 = 0. \tag{3}$$

Moreover, if u^* is a continuous function of t then

$$\dot{p}_1 = -p_0 c f / x_1^2 + p_0 c_0 (1 - F(t_1)) - p_2 \tag{4}$$
$$\dot{p}_2 = p_0 c_0 f. \tag{5}$$

Conditions (2),(3) and (5) characterize the analytical form of the adjoint function p_2 in the following way

$$p_2(t) = p_2(t_1) + \int_{t_1}^{t} c_0 f(\tau) d\tau = c_0[F(t) - F(t_1)]$$

so, the previous system may be rewritten in the equivalent form

$$\dot{p}_1(t) = c_0(1 - F(t)) - cf(t)/x_1^2(t) \tag{6}$$
$$p_1(t_1) = 0$$

The Pontryagin's Maximum Principle gives also information on the analytical form of the optimal control $u^*(t)$: in fact

$$u^*(t) = \begin{cases} \overline{u}, & \text{if } p_1(t) > 0 \\ 0, & \text{if } p_1(t) < 0 \end{cases}$$

and if $u^*(t)$ is continuous on t then u^* satisfies the conditions (6) and (1).

The literature on random checking schedules presents results in which existence of optimal solutions is proved, both for the original problem both for the approximation of the original problem. Nevertheless, the characterization of optimal or sub-optimal solutions is made by using non-linear programming conditions, such as Pontryagin's Maximun Principle, which are usually difficult to solve: in order to actually determine the optimal solutions it is often necessary to use numerical techniques. In our work we refer to particular classes of failure rate distributions in order to obtain the analytical solution to the problem of detecting sub-optimal audit scheme for system subject to random control.

3 Sub-Optimal audit scheme: the case of exponential failure density function

Using the same notations as in the previous section, the system of differential equations associated to the *RAS* problem is

$$\begin{cases} \dot{x}_1(t) = u(t) \\ \dot{p}_1(t) = c_0(1 - F(t)) - cf(t)/x_1^2(t) \\ x_1(0) = a \\ p_1(t_1) = 0 \end{cases} \tag{7}$$

Let us suppose that the failure density function is exponential, that is

$$f(t) = \lambda \exp(-\lambda t), \qquad \lambda > 0. \tag{8}$$

If u^* is continuous at t, then the adjoint function $p_1 = p_1(t)$ satisfies the following set of conditions

$$\begin{aligned} \dot{p}_1(t) &= \exp(-\lambda t)[c_0 - c\lambda/x_1^2(t)] \\ p_1(t_1) &= 0. \end{aligned} \tag{9}$$

Moreover,

$$\dot{p}_1(t) \geq 0 \qquad \Longleftrightarrow \qquad [x_1(t)]^{-2} \leq c_0/(c\lambda)$$

so the following exhaustive cases are possible:

i) $1/a^2 < c_0/(c\lambda)$;
ii) $1/a^2 = c_0/(c\lambda)$;
iii) $[x_1(t)]^{-2} > c_0/(c\lambda)$;
iv) $[x_1(t)]^{-2} = c_0/(c\lambda)$ for each $t \in [\bar{t}, t_1]$, where $t = [(c\lambda/c_0)^{1/2} - a]/\bar{u}$.

In this way it is possible to determine the optimal control u^* and the optimal intensity of the Poisson control process $x_1^*(t)$ of the *RAS* problem; in fact

$$x_1^*(t) = a + \int_0^t u^*(\tau) d\tau.$$

The following proposition characterizes some aspects of the functions $x_1 = x_1^*(t)$ and $p_1 = p_1(t)$, in the phase diagram X_1P_1.

Theorem 1 *Let a system be subject to random failure described by probability density function $f(t) = \lambda \exp(-\lambda t)$, $(\lambda > 0)$. In the phase diagram X_1P_1, $p_{10} = p_1(0)$ is a decreasing convex function of $x_{10} = x_1^*(0)$. In the region where $x_1 < (c\lambda/c_0)^{1/2}$, $p_1 = p_1(t)$ is a decreasing convex function of $x_1 = x_1^*(t)$.*

Proof

If $1/a^2 < c_0/(c\lambda)$, then (for more details, see the Appendix)

$$p_1(t) = (c_0 - c\lambda/a^2)[\exp(-\lambda t)]/\lambda \leq 0.$$

Clearly, p_{10} is a decreasing convex function of $x_{10} = a$, in fact

$$\frac{dp_{10}}{dx_{10}} = 2c/x_{10}^3[\exp(-\lambda t_1) - 1] < 0$$

and

$$\frac{d^2p_{10}}{dx_{10}{}^2} = -6c/x_{10}^4[\exp(-\lambda t_1) - 1] > 0.$$

When $1/a^2 = c_0/(c\lambda)$ it follows $p_1(t) = 0$ and $x_1(t) = (c\lambda/c_0)^{1/2}$. If $[x_1(t)]^{-2} > c_0/(c\lambda)$ then (see the Appendix)

$$p_1(t) = -\int_t^{t_1} \exp(-\lambda t)[c_0 - c\lambda/x_1^2(\tau)]d\tau \geq 0,$$

by which we obtain

$$\frac{dp_{10}}{dx_{10}} = \int_0^{t_1} \exp(-\lambda t)[-2c\lambda/(x_{10} + \bar{u}\tau)^3]d\tau < 0$$

and

$$\frac{d^2p_{10}}{dx_{10}{}^2} = \int_0^{t_1} \exp(-\lambda t)[6c\lambda/(x_{10} + \bar{u}\tau)^4]d\tau > 0.$$

In this case, because $x_1(t) = a + \bar{u}t$ is a monotone function and $\dot{p}_1 < 0$, $p_1(t)$ is a decreasing function of x_1. Moreover, it is a convex function, in fact

$$\frac{d^2p_1}{dx_1{}^2} = \exp[-\lambda(x_1 - a)/\bar{u}][2c/x_1^3 - (c_0 - c\lambda/x_1^2)/\bar{u}]\lambda/\bar{u} > 0. \tag{10}$$

Finally, if $[x_1(t)]^{-2} = c_0/(c\lambda)$ for each $t \in [\bar{t}, t_1]$, where $t = [(c\lambda/c_0)^{1/2} - a]/\bar{u}$ then (see the Appendix)

$$p_1(t) = \begin{cases} \int_t^{\bar{t}} \exp(-\lambda t)[c\lambda/(a + \bar{u}\tau)^2 - c_0]d\tau & \text{if } 0 \leq t \leq \bar{t}, \\ 0 & \text{otherwise.} \end{cases}$$

The function p_1 is a decreasing function of x_1 in $[0, t_1]$, in fact $\dot{p}_1(t) < 0$ in $[0, \bar{t}]$ and x_1 is an increasing function of t. Moreover p_1 is a convex function of x_1, in fact for every $t \in [0, \bar{t})$ we have (10). In the same way, p_{10} is a decreasing convex function of x_{10}, in fact

$$\frac{dp_{10}}{dx_{10}} = \int_0^{\bar{t}} \exp(-\lambda t)[-2c\lambda/(x_{10} + \bar{u}\tau)^3]d\tau \leq 0$$

and

$$\frac{d^2p_{10}}{dx_{10}{}^2} = \int_0^{\bar{t}} \exp(-\lambda t)[6c\lambda/(x_{10} + \bar{u}\tau)^4]d\tau \geq 0.$$

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4 Sub-Optimal audit scheme: the case of uniform failure density function

If we suppose that the failure density function is uniform, that is

$$f(t) = \lambda, \qquad \lambda > 0. \tag{11}$$

In this case it is necessary to guarantee that the intensity λ and the reference time t_1 satisfy the condition $t_1 \leq 1/\lambda$. In that case the adjoint function $p_1 = p_1(t)$ satisfies the following set of conditions

$$\dot{p}_1(t) = c_0(1 - \lambda t) - c\lambda/x_1^2(t)] \tag{12}$$

$$p_1(t_1) = 0. \tag{13}$$

with the exception of the points t where u^* is not continuous. Let us call

$$\phi(t) = c\lambda/c_0(1 - \lambda t)^{-1}.$$

Clearly,

$$\dot{p}_1(t) \geq 0 \qquad \Longleftrightarrow \qquad x_1^2(t) \geq \phi(t).$$

It is possible to characterize the function $p_1 = p_1(t)$ satisfying (12) and (13), and the corresponding functions u^* and x_1^* by analyzing these exhaustive cases:

i) $a^2 \geq \phi(t_1)$;
ii) $a^2 < \phi(t_1)$ and there is no instant $\tilde{t}$, $0 \leq \tilde{t} < t_1$, such that $\tilde{t} = \inf\{t : p_1(t) > 0\}$;
iii) $a^2 < \phi(t_1)$ and there is an instant $\tilde{t}$, $0 \leq \tilde{t} < t_1$, such that $\tilde{t} = \inf\{t : p_1(t) > 0\}$;

The Appendix contains the detailed analysis of each case and presents the related results. The following proposition sets an analytic link between $p_{10} = p_1(0)$ and $x_{10} = x_1(0)$.

Theorem 2 *Let a system be subject to random failure described by probability density function $f(t) = \lambda$, ($\lambda > 0$). In the phase diagram X_1P_1, $p_{10} = p_1(0)$ is a decreasing convex function of $x_{10} = x_1^*(0)$.*

Proof

If $a^2 > \phi(t_1)$ then

$$p_1(t) = c_0\lambda/2(t_1^2 - t^2) + (c\lambda/a^2 - c_0)(t_1 - t) \leq 0.$$

Monotonicity and convexity of p_{10} with respect to $x_{10} = a$ follow by

$$\frac{dp_{10}}{dx_{10}} = -2c\lambda t_1/x_{10}^3 < 0$$

and

$$\frac{d^2 p_{10}}{d{x_{10}}^2} = 6c\lambda t_1 x_{10}^4 > 0.$$

Note that if $a^2 = \phi(t_1) = c\lambda/c_0(1-\lambda t_1)^{-1}$ then

$$p_1(t) = -c_0\lambda/2(t_1 - t)^2,$$

and

$$p_1(0) = -c_0\lambda t_1^2/2.$$

If $a^2 < \phi(t_1)$ and there is no instant $\tilde{t}$, $0 \le \tilde{t} < t_1$, such that $\tilde{t} = \inf\{t : p_1(t) > 0\}$, then

$$p_1(t) = c\lambda/\bar{u}[1/(a + \bar{u}t) - 1/(a + \bar{u}t_1)] + c_0\lambda/2(t_1^2 - t^2) - c_0(t_1 - t).$$

It follows

$$\frac{dp_{10}}{dx_{10}} = c\lambda/\bar{u}[1/(x_{10} + \bar{u}t_1)^2 - 1/(x_{10}^2 < 0$$

and

$$\frac{d^2 p_{10}}{d{x_{10}}^2} = c\lambda/\bar{u}[2/x_{10}^3 - 2/(x_{10} + \bar{u}t_1)^3 > 0.$$

Finally, if $a^2 < \phi(t_1)$ and there exists an instant $\tilde{t}$, $0 \le \tilde{t} < t_1$, such that $\tilde{t} = \inf\{t : p_1(t) > 0\}$, we have

$$p_1(t) = \begin{cases} c_0\lambda(\tilde{t}^2 - t^2)/2 + (c\lambda/a^2 - c_0)(\tilde{t} - t) & \text{if } t \in [0, \tilde{t}), \\ c\lambda/\bar{u}\{1/[a + \bar{u}(t - \tilde{t})] - 1/[a + \bar{u}(t_1 - \tilde{t})]\} + c_0\lambda(t_1^2 - t^2)/2 - c_0(t_1 - t) & \text{otherwise.} \end{cases}$$

In particular,

$$p_1(0) = c_0\lambda\tilde{t}^2 + (c\lambda/a^2 - c_0)\tilde{t},$$

from which we have

$$\frac{dp_{10}}{dx_{10}} = -2c\lambda\tilde{t}/x_{10}^3 < 0,$$

and

$$\frac{d^2 p_{10}}{d{x_{10}}^2} = 6c\lambda\tilde{t}/x_{10}^4 > 0.$$

■

The case of uniform failure density function has associated a system of differential equations calling for the study of a difficult equation: in fact the integral condition characterizing the instant $\tilde{t}$ (see the Appendix)

$$\int_{\tilde{t}}^{t_1} \{c_0(1 - \lambda t) - c\lambda/[a + \bar{u}(t - \tilde{t})]^2\} dt$$

requires numerical solution algorithms. In addition to this it is also an open question to determine the correct direction of the inequality involving $\phi(t_1)$ and $(2c\bar{u}/c_0)^{1/3}$. Anyway, it is possible to characterize the behaviour of the functions

$x_1 = x_1^*(t)$ and $p_1 = p_1(t)$ in the phase diagram X_1P_1, in relation of the possible solutions of the previous questions. In fact, by the analysis of the case *ii*) it results that p_1 is a decreasing function of x_1: this is true because $x_1 = a + \bar{u}t$ is a monotone function and $\dot{p}_1 < 0$. Moreover, because

$$\frac{d^2p_1}{dx_1{}^2} = \lambda/\bar{u}(2c/x_1^3 - c_0/\bar{u}), \tag{14}$$

we have that p_1 is a convex function of x_1 if and only if

$$x_1 < (2c\bar{u}/c_0)^{1/3}.$$

The case *iii*) establishes that in $[\tilde{t}, t_1]$ the function p_1 is first increasing and then decreasing in x_1. In fact

$$\frac{dp_1}{dx_1} = c_0/\bar{u}[1 - \lambda\tilde{t} - \lambda(x_1 - a)/\bar{u}] - c\lambda/\bar{u}x_1^2.$$

Furthermore, it is a convex function of x_1 if and only if

$$x_1 < (2c\bar{u}/c_0)^{1/3},$$

as (14) is also true.

5 Conclusion

Measuring and monitoring environmental and system performance, together with reviewing, evaluating and improving the system represent part of the activities included by the EMAS system. Our aim is detecting an optimal sequence of random inspections in such a way the EMAS audit schemes can better achieve the assessment of the management system and the conformity with the environmental policy and programme.

6 Appendix

This Section is devoted to the study of the solutions of the system of differential equations associated to the *RAS* problem:

$$\begin{cases} \dot{x}_1(t) = u(t) \\ \dot{p}_1(t) = c_0(1 - F(t)) - cf(t)/x_1^2(t) \\ x_1(0) = a \\ p_1(t_1) = 0 \end{cases}$$

6.1 The case of exponential failure density function

In the following analysis four mutually exclusive cases are studied.

i) $1/a^2 < c_0/(c\lambda)$.
Because $x_1(t)^{-2} < c_0/(c\lambda)$ on $[0, t_1]$ and $p_1(t_1) = 0$, we have that $p_1(t) < 0$ for every $t \in [0, t_1]$, then

$$u^*(t) = 0$$

$$x_1^*(t) = a$$

on $[0, t_1]$.

ii) $1/a^2 = c_0/(c\lambda)$.
Let $t_2 = \sup\{t : x_1^{-2}(t) = c_0/(c\lambda)\}$. Necessarily, $t_2 \geq t_1$. Then

$$\begin{aligned} &u^*(t) = 0, \\ &x_1^*(t) = a = (c\lambda/c_0)^{1/2} \end{aligned}$$

on $[0, t_1]$.

iii) $[x_1(t)]^{-2} > c_0/(c\lambda)$.
Condition (1) ensures positivity of p_1 on $[0, t_1)$, so we have

$$\begin{aligned} &u^*(t) = 0, \\ &x_1^*(t) = a + \bar{u}t \end{aligned}$$

on $[0, t_1]$.

iv) $[x_1(t)]^{-2} = c_0/(c\lambda)$ for each $t \in [\bar{t}, t_1]$, where $t = [(c\lambda/c_0)^{1/2} - a]/\bar{u}$. In this case

$$\dot{p}_1 \begin{cases} < 0, & \text{if } 0 \leq t < \bar{t} \\ = 0, & \text{if } \bar{t} \leq t \leq t_1 \end{cases}.$$

It follows that

$$p_1(t) \begin{cases} > 0, & \text{if } 0 \leq t < \bar{t} \\ = 0, & \text{if } \bar{t} \leq t \leq t_1 \end{cases}.$$

Clearly,

$$\begin{aligned} &u^*(t) = \bar{u}, \qquad t \in [0, \bar{t}] \\ &x_1^*(t) = \begin{cases} a + \bar{u}t, & \text{if } 0 \leq t < \bar{t} \\ a + \bar{u}\bar{t}, & \text{if } \bar{t} \leq t \leq t_1. \end{cases} \end{aligned}$$

Because $x_1^{-2}(t) = c_0/(c\lambda)$, it ensues that

$$\bar{t} = [(c\lambda/c_0)^{1/2} - a]/\bar{u}.$$

The definition of the optimal control intensity is

$$x_1^*(t) = \begin{cases} a + \bar{u}t, & \text{if } 0 \leq t < \bar{t} \\ (c\lambda/c_0)^{1/2}, & \text{if } \bar{t} \leq t \leq t_1 \end{cases}.$$

6.2 The case of uniform failure density function

The analysis consists of the study of three cases.

i) $a^2 \geq \phi(t_1)$.
By increasing monotonicity of $x_1(t)$ and $\phi(t)$, and condition $x_1(0) = a$ it follows that $p_1(t) < 0$ for every $t \in [0, t_1)$. Then

$$u^*(t) = 0,$$
$$x_1^*(t) = a$$

on $[0, t_1]$. Note that $t_1 < 1/\lambda$.

ii) $a^2 < \phi(t_1)$ and there is no instant $\tilde{t}$, $0 \leq \tilde{t} < t_1$, such that $\tilde{t} = \inf\{t : p_1(t) > 0\}$.
The state function $x_1(t)$ is increasing. Let t_2 be such that

$$t_2 = \inf\{t : x_1^2(t) < \phi(t)\}.$$

Then

$$x_1^2(t) < \phi(t)$$

for every $t \in [t_2, t_1]$. The condition characterizing this case ensures $p_1(t) > 0$ for every $t \in [0, t_1)$. So

$$u^*(t) = \bar{u},$$
$$x_1^*(t) = a + \bar{u}t$$

on $[0, t_1]$. In this case $t_1 \leq 1/\lambda$.

iii) $a^2 < \phi(t_1)$ and there is an instant $\tilde{t}$, $0 \leq \tilde{t} < t_1$, such that $\tilde{t} = \inf\{t : p_1(t) > 0\}$.
With an equivalent analysis it is possible to deduce that

$$\dot{p}_1 \begin{cases} < 0, & \text{if } 0 \leq t < \tilde{t} \\ > 0, & \text{if } \tilde{t} < t < t_1 \end{cases}.$$

The optimal control is

$$u^*(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \tilde{t} \\ \bar{u}, & \text{if } \tilde{t} < t \leq t_1 \end{cases}.$$

Moreover,

$$x_1^*(t) = \begin{cases} a, & \text{if } 0 \leq t \leq \tilde{t} \\ a + \bar{u}(t - \tilde{t}), & \text{if } \tilde{t} < t \leq t_1 \end{cases}.$$

The condition $p_1(\tilde{t}) = 0$ characterizes the instant $\tilde{t}$ as follows

$$\int_{\tilde{t}}^{t_1} c_0(1 - \lambda t) - c\lambda/[a + \bar{u}(t - \tilde{t})]^2 dt.$$

Note that in this case $t_1 \leq 1/\lambda$.

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